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Certain sufficient conditions for a subclass of analytic functions involving Hohlov operator

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ABSTRACT

Making use of the Hohlov operator, the authors obtain inclusion relations between the classes of certain normalized analytic functions. Relevant connections of our work with the earlier works are pointed out.

1. Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}).$$

This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that are starlike in $\mathbb{U}$ with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order $\alpha (0 \leq \alpha < 1)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}).$$

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This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. It is an established fact that

$$f \in \mathcal{K}(\alpha) \quad \Longleftrightarrow \quad zf' \in \mathcal{S}^*(\alpha).$$

Let $M(\lambda, \alpha)$ be a subclass of $\mathcal{A}$ consisting of functions of the form that satisfy the condition

$$\Re\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right) > \alpha, \quad z \in \Delta$$

for some α and λ where $0 \le \alpha < 1$ and $0 \le \lambda < 1$. The class $M(\lambda, \alpha)$ was introduced by Altintas and Owa [1] and also investigated very recently by Mostafa [2].

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}$ of uniformly convex functions in $\mathbb{U}$ if and only if it has the property that, for every circular arc δ contained in the unit disk $\mathbb{U}$, with center ζ also in $\mathbb{U}$, the image curve $f(\delta)$ is a convex arc. The function class $\mathcal{UCV}$ was introduced by Goodman [3].

Furthermore, we denote by $k - \mathcal{UCV}$ and $k - \mathcal{ST}$, $(0 \le k < \infty)$, two interesting subclasses of $\mathcal{S}$ consisting respectively of functions which are k-uniformly convex and k-starlike in $\mathbb{U}$. Namely, we have for $0 \le k < \infty$

$$k - \mathcal{UCV} := \left\{ f \in \mathcal{S} \colon \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, (z \in \mathbb{U}) \right\}$$

and

$$k - \mathcal{ST} := \left\{ f \in \mathcal{S} \colon \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, (z \in \mathbb{U}) \right\}.$$

The class $k - \mathcal{UCV}$ was introduced by Kanas and Wiśniowska [4], where its geometric definition and connections with the conic domains were considered. The class $k - \mathcal{ST}$ was investigated in [5]. In fact, it is related to the class $k - \mathcal{UCV}$ by means of the well-known Alexander equivalence between the usual classes of convex and starlike functions (see also the work of Kanas and Srivastava [6] for further developments involving each of the classes $k - \mathcal{UCV}$ and $k - \mathcal{ST}$). In particular, when $k = 1$, we obtain

$$k - \mathcal{UCV} \equiv \mathcal{UCV} \quad \text{and} \quad k - \mathcal{ST} = \mathcal{SP},$$

where $\mathcal{UCV}$ and $\mathcal{SP}$ are the familiar classes of uniformly convex functions and parabolic starlike functions in $\mathbb{U}$ respectively (see for details, [3]). Indeed, by making use of a certain fractional calculus operator, Srivastava and Mishra [7] presented a systematic and unified study of the classes $\mathcal{UCV}$ and $\mathcal{SP}$.

Let us denote (see [4,5])

$$P_1(k) = \begin{cases} \dfrac{8(\arccos k)^2}{\pi^2(1-k^2)} & \text{for } 0 \le k < 1 \\ \dfrac{8}{\pi^2} & \text{for } k = 1 \\ \dfrac{\pi^2}{4\sqrt{t}(1+t)(k^2-1)\mathcal{K}^2(t)} & \text{for } k > 1, \end{cases} \tag{1.2}$$

where $t \in (0, 1)$ is determined by $k = \cosh(\pi\mathcal{K}'(t)/[4\mathcal{K}(t)])$, $\mathcal{K}$ is the Legendre's complete Elliptic integral of the first kind

$$\mathcal{K}(t) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

and $\mathcal{K}'(t) = \mathcal{K}(\sqrt{1-t^2})$ is the complementary integral of $\mathcal{K}(t)$. Let Ω_k be a domain such that $1 \in \Omega_k$ and

$$\partial\Omega_k = \left\{ w = u + iv \colon u^2 = k^2(u-1)^2 + k^2v^2 \right\}, \quad 0 \le k < \infty.$$

The domain Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic when $k = 1$, and a right half-plane when $k = 0$. If p is an analytic function with $p(0) = 1$ which maps the unit disk $\mathbb{U}$ conformally onto the region Ω_k, then $P_1(k) = p'(0)$. $P_1(k)$ is strictly decreasing function of the variable k and it values are included in the interval $(0, 2]$.

Let $f \in \mathcal{A}$ be of the form (1.1). If $f \in k - \mathcal{UCV}$, then the following coefficient inequalities hold true (cf. [4]):

$$|a_n| \le \frac{(P_1(k))_{n-1}}{n!}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{1.3}$$

Similarly, if f of the form (1.1) belongs to the class $k - \mathcal{ST}$, then (cf., [5])

$$|a_n| \le \frac{(P_1(k))_{n-1}}{(n-1)!}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{1.4}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A, B)$, $(\tau \in \mathbb{C} \setminus \{0\}, -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1 \quad (z \in \mathbb{U}).$$

The class $\mathcal{R}^{\tau}(A, B)$ was introduced earlier by Dixit and Pal [8]. Two of the many interesting subclasses of the class $\mathcal{R}^{\tau}(A, B)$ are worthy of mention here. First of all, by setting

$$\tau = e^{i\eta} \cos\eta \ (-\pi/2 < \eta < \pi/2), \qquad A = 1 - 2\beta \ (0 \leq \beta < 1) \quad \text{and} \quad B = -1,$$

the class $\mathcal{R}^{\tau}(A, B)$ reduces essentially to the class $\mathcal{R}_{\eta}(\beta)$ introduced and studied by Ponnusamy and Rønning [9], where

$$R_{\eta}(\beta) = \left\{ f \in \mathcal{A}\colon \ \Re(e^{i\eta}(f'(z) - \beta)) > 0 \ (z \in \mathbb{U}; -\pi/2 < \eta < \pi/2, 0 \leq \beta < 1) \right\}.$$

Secondly, if we put

$$\tau = 1, \qquad A = \beta \quad \text{and} \quad B = -\beta \ (0 < \beta \leq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in \mathbb{U}; 0 < \beta \leq 1)$$

which was studied by (among others) Padmanabhan [10] and Caplinger and Causey [11], (see the works of [12–15] also).

The Gaussian hypergeometric function $F(a, b; c; z)$ given by

$${}_2F_1(a, b; c; z) = F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in \mathbb{U}) \tag{1.5}$$

is the solution of the homogeneous hypergeometric differential equation

$$z(1 - z)w''(z) + [c - (a + b + 1)z]w'(z) - abw(z) = 0$$

and has rich applications in various fields such as conformal mappings, quasi conformal theory, continued fractions and so on.

By the Gauss Summation theorem, we get,

$$F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n \, (b)_n}{(c)_n (1)_n} = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - a)\Gamma(c - b)} \quad \text{for } \Re(c - a - b) > 0.$$

Here, a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, \ldots$, $(a)_0 = 1$ for $a \neq 0$, and for each positive integer n, $(a)_n = a(a + 1)(a + 2) \cdots (a + n - 1)$ is the Pochhammer symbol. In the case of $c = -k$, $k = 0, 1, 2, \ldots$, $F(a, b; c; z)$ is defined if $a = -j$ or $b = -j$ where $j \leq k$. In this situation, $F(a, b; c; z)$ becomes a polynomial of degree j with respect to z. Results regarding $F(a, b; c; z)$ when $\Re\,(c - a - b)$ is positive, zero or negative are abundant in the literature. In particular when $\Re\,(c - a - b) > 0$, the function is bounded. The hypergeometric function $F(a, b; c; z)$ has been studied extensively by various authors and play an important role in Geometric Function Theory. It is useful in unifying various functions by giving appropriate values to the parameters a, b and c. We refer to [16,12–14] and references therein for some important results.

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.6}$$

For $f \in \mathcal{A}$, we recall the operator $I_{a,b,c}(f)$ of Hohlov [17] which maps $\mathcal{A}$ into itself defined by means of Hadamard product as

$$I_{a,b,c}(f)(z) = zF(a, b; c; z) * f(z). \tag{1.7}$$

Therefore, for a function f defined by (1.1), we have

$$I_{a,b,c}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} a_n z^n. \tag{1.8}$$

Using the integral representation,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1} \frac{\mathrm{d}t}{(1 - tz)^a}, \quad \Re(c) > \Re(b) > 0,$$

we can write

$$[I_{a,b,c}(f)](z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\frac{f(tz)}{t}\mathrm{d}t*\frac{z}{(1-tz)^a}.$$

When $f(z)$ equals the convex function $\frac{z}{1-z}$, then the operator $I_{a,b,c}(f)$ in this case becomes $zF(a,b;c;z)$. If $a=1$, $b=1+\delta$, $c=2+\delta$ with $\Re(\delta)>-1$ then the convolution operator $I_{a,b,c}(f)$ turns into Bernardi operator

$$B_f(z)=[I_{a,b,c}(f)](z)=\frac{1+\delta}{z^\delta}\int_0^1 t^{\delta-1}f(t)\mathrm{d}t.$$

Indeed, $I_{1,1,2}(f)$ and $I_{1,2,3}(f)$ are known as Alexander and Libera operators, respectively.

To prove the main results, we need the following Lemmas.

Lemma 1 ([1]). *A function $f\in\mathcal{A}$ belongs to the class $M(\lambda,\alpha)$ if*

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)|a_n|\le 1-\alpha. \tag{1.9}$$

Lemma 2 ([8]). *If $f\in\mathcal{R}^\tau(A,B)$ is of form (1.1), then*

$$|a_n|\le (A-B)\frac{|\tau|}{n},\quad n\in\mathbb{N}\setminus\{1\}. \tag{1.10}$$

The result is sharp.

In this paper, we estimate certain inclusion relations involving the classes $k-\mathcal{UCV}$, $k-\mathcal{ST}$ and $M(\lambda,\alpha)$.

2. Main results

Theorem 1. *Let $a,b\in\mathbb{C}\setminus\{0\}$. Also, let $|a|\neq 1$, $b\neq 1$ and c be a real number such that $c>|a|+|b|+1$. If $f\in\mathcal{R}^\tau(A,B)$, and if the inequality*

$$\begin{aligned}&\frac{\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(1-\lambda\alpha)(c-|a|-|b|-1)+\frac{\alpha(\lambda-1)}{(|a|-1)(|b|-1)}\right]\\ &\quad\le(1-\alpha)\left(\frac{1}{(A-B)|\tau|}+1\right)+\alpha(\lambda-1)\frac{c-1}{(|a|-1)(|b|-1)}\end{aligned} \tag{2.1}$$

is satisfied, then $I_{a,b,c}(f)\in M(\lambda,\alpha)$.

Proof. Let f be of the form (1.1) belong to the class $R^\tau(A,B)$. By virtue of Lemma 1, it suffices to show that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|\le 1-\alpha. \tag{2.2}$$

Taking into account the inequality (1.10) and the relation $|(a)_{n-1}|\le(|a|)_{n-1}$, we deduce that

$$\begin{aligned}&\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|\\ &\quad\le(A-B)|\tau|(1-\lambda\alpha)\sum_{n=2}^{\infty}\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|+(A-B)|\tau|\alpha(\lambda-1)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}}\\ &\quad\le(A-B)|\tau|(1-\lambda\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}+(A-B)|\tau|\alpha(\lambda-1)\frac{(c-1)}{(|a|-1)(|b|-1)}\sum_{n=2}^{\infty}\frac{(|a|-1)_n(|b|-1)_n}{(c-1)_n(1)_n}\\ &\quad=(1-\lambda\alpha)(A-B)|\tau|\left(F(|a|,|b|;c;1)-1\right)\\ &\qquad+(A-B)|\tau|\alpha(\lambda-1)\frac{(c-1)}{(|a|-1)(|b|-1)}\left(F(|a|-1,|b|-1;c-1;1)-\frac{(|a|-1)(|b|-1)}{c-1}-1\right)\end{aligned}$$

where we use the relation

$$(a)_n=a(a+1)_{n-1}. \tag{2.3}$$

The proof now follows by an application of Gauss summation theorem and (2.1). □

For the choice of $|b| = |a|$, we have the following corollary.

Corollary 1. *Let $a \in \mathbb{C} \setminus \{0\}$, and $|a| \neq 1$. Also, let c be a real number such that $c > 2|a| + 1$. If $f \in \mathcal{R}^{\tau}(A, B)$, and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-2|a|-1)}{(\Gamma(c-|a|))^2}\left[(1-\lambda\alpha)(c-2|a|-1)+\frac{\alpha(\lambda-1)}{(|a|-1)^2}\right] \leq (1-\alpha)\left(\frac{1}{(A-B)|\tau|}+1\right)+\alpha(\lambda-1)\frac{c-1}{(|a|-1)^2} \tag{2.4}$$

is satisfied, then $I_{a,\,|a|,\,c}(f) \in M(\lambda, \alpha)$.

Theorem 2. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + 2$. If $f \in \mathcal{S}$ and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1-\alpha+\frac{(1-\lambda\alpha)(a)_2(b)_2}{(1-\alpha)(c-|a|-|b|-2)_2} + (3-2\lambda\alpha-\alpha)\frac{|ab|}{c-|a|-|b|-1}(c-|a|-|b|-1)\right] \leq 2(1-\alpha) \tag{2.5}$$

is satisfied, then $I_{a,\,b,\,c}(f) \in M(\lambda, \alpha)$.

Proof. Let f be of the form (1.1) belong to the class $\mathcal{S}$. Applying the well known estimate for the coefficients of the functions $f \in \mathcal{S}$, due to de Branges [18], we need to show that

$$\sum_{n=2}^{\infty} n(n-\lambda\alpha n-\alpha+\lambda\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq 1-\alpha. \tag{2.6}$$

Taking into account the inequality $|(a)_{n-1}| \leq (|a|)_{n-1}$, we deduce that

$$S(a,b,c,\lambda,\alpha) \leq \sum_{n=2}^{\infty}\left(n^2(1-\lambda\alpha)+n\alpha(\lambda-1)\right)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

writing $n = (n-1)+1$, and $n^2 = (n-1)(n-2)+3(n-1)+2$, we can rewrite the above term as

$$S(a,b,c,\lambda,\alpha) \leq (1-\lambda\alpha)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + (3-2\lambda\alpha-\alpha)\sum_{n=2}^{\infty}(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + (1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}.$$

Repeatedly using the relation given in (2.3),

$$S(a,b,c,\lambda,\alpha) \leq (1-\lambda\alpha)\frac{(a)_2(b)_2}{(c)_2}\sum_{n=3}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}} + (3-2\lambda\alpha-\alpha)\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} + (1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}.$$

The inequality (2.6) now follows by applying Gauss summation theorem and (2.5). □

Repeating the above reasoning for $|b| = |a|$ we can improve the assertion of Theorem 2 as follows.

Corollary 2. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > 2|a| + 2$. If $f \in \mathcal{S}$ and if the inequality*

$$\frac{\Gamma(c)\,\Gamma(c-2|a|-1)}{(\Gamma(c-|a|))^2}\left[1-\alpha+\frac{(1-\lambda\alpha)((a)_2)^2}{(1-\alpha)(c-2|a|-2)_2}+(3-2\lambda\alpha-\alpha)\frac{|a|^2}{c-2|a|-1}(c-2|a|-1)\right] \leq 2(1-\alpha)$$

is satisfied, then $I_{a,\,|a|,\,c}(f) \in M(\lambda, \alpha)$.

In the special case when $b = 1$, Theorem 2 immediately yields a result concerning the Carlson–Shaffer operator $\mathcal{L}(a, c)(f) := I_{a,\,1,\,c}(f)$ (see [16]).

Corollary 3. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + 3$. If $f \in \mathcal{S}$ and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-2)}{\Gamma(c-|a|)\Gamma(c-1)}\left[1-\alpha+\frac{2(1-\lambda\alpha)(a)_2}{(1-\alpha)(c-|a|-3)_2}+(3-2\lambda\alpha-\alpha)\frac{|ab|}{c-|a|-2}(c-|a|-2)\right] \\ \leq 2(1-\alpha),$$

is satisfied, then $\mathcal{L}(a, c)f \in M(\lambda, \alpha)$.

Theorem 3. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + P_1$ and $P_1 = P_1(k)$ be given by (1.2). If $f \in k - \mathcal{UCV}$, for some k $(0 \leq k < \infty)$, and the inequality*

$$(1-\lambda\alpha)\ {}_3F_2(|a|, |b|, P_1; c, 1; 1) + \alpha(\lambda-1)\ {}_3F_2(|a|, |b|, P_1; c, 2; 1) \leq 2(1-\alpha) \tag{2.7}$$

is satisfied, then $I_{a,b,c}(f) \in M(\lambda, \alpha)$.

Proof. Let f given by (1.1) belong to $k - \mathcal{UCV}$. By (1.9), to show $I_{a,b,c}(f) \in M(\lambda, \alpha)$, it is sufficient to prove that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leq 1-\alpha. \tag{2.8}$$

We will repeat the method of proving used in the proof of Theorem 1. Applying the estimates for the coefficients given by (1.3), and making use of the relations (2.3) and $|(a)_n| \leq (|a|)_n$, we get

$$\begin{aligned}\sum_{n=2}^{\infty}&(n-\lambda\alpha n-\alpha+\lambda\alpha)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \\ &\leq \sum_{n=2}^{\infty}[n(1-\lambda\alpha)+\alpha(\lambda-1)]\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_n} \\ &= (1-\lambda\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}+\alpha(\lambda-1)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_n} \\ &= (1-\lambda\alpha)\left[{}_3F_2(|a|, |b|, P_1; c, 1; 1)-1\right]+\alpha(\lambda-1)\ \left[{}_3F_2(|a|, |b|, P_1; c, 2; 1)-1\right] \leq 1-\alpha\end{aligned}$$

provided the condition (2.7) is satisfied. □

If $|b| = |a|$ we can rewrite the Theorem 3 as follows.

Corollary 4. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > 2|a| + P_1$ and $P_1 = P_1(k)$ be given by (1.2). If $f \in k - \mathcal{UCV}$ for some k $(0 \leq k < \infty)$ and the inequality*

$$(1-\lambda\alpha){}_3F_2(|a|, |a|, P_1; c, 1; 1) + \alpha(\lambda-1)\ {}_3F_2(|a|, |a|, P_1; c, 2; 1) \leq 2(1-\alpha) \tag{2.9}$$

is satisfied, then $I_{a,|a|,c}(f) \in M(\lambda, \alpha)$.

In the special case when $b = 1$, Theorem 3 immediately yields a result concerning the Carlson–Shaffer operator $\mathcal{L}(a, c)(f) = I_{a,1,c}(f)$.

Corollary 5. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > 1 + |a| + P_1$ and $P_1 = P_1(k)$ be given by (1.2). If $f \in k - \mathcal{UCV}$ for some k $(0 \leq k < \infty)$ and the inequality*

$$(1-\lambda\alpha)\ {}_3F_2(|a|, 1, P_1; c, 1; 1) + \alpha(\lambda-1)\ {}_3F_2(|a|, 1, P_1; c, 2; 1) \leq 2(1-\alpha) \tag{2.10}$$

is satisfied, then $\mathcal{L}(a, c)(f) \in M(\lambda, \alpha)$.

Theorem 4. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + P_1 + 1$ and $P_1 = P_1(k)$ be given by (1.2). If, for some k $(0 \leq k < \infty)$, $f \in k - \mathcal{ST}$, and the inequality*

$$(1-\lambda\alpha)\frac{|ab|P_1}{c}\ {}_3F_2(1+|a|, 1+|b|, 1+P_1; 1+c, 2; 1) + (1-\alpha)\ {}_3F_2(|a|, |b|, P_1; c, 1; 1) \leq 2(1-\alpha) \tag{2.11}$$

is satisfied, then $I_{a,b,c}(f) \in M(\lambda, \alpha)$.

Proof. Let f be given by (1.1) belong to $k - \mathcal{ST}$. Applying the estimates for the coefficients given by (1.4), and making use of the relation (2.3) and $|(a)_n| \leq (|a|)_n$, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n - \lambda\alpha n - \alpha + \lambda\alpha)\ \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \\
&\quad \leq \sum_{n=2}^{\infty}[n(1-\lambda\alpha) + \alpha(\lambda - 1)]\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
&\quad = (1-\lambda\alpha)\sum_{n=2}^{\infty}\frac{|ab|P_1}{c}\frac{(1+|a|)_{n-2}(1+|b|)_{n-2}(1+P_1)_{n-2}}{(1+c)_{n-2}(1)_{n-2}(2)_{n-2}} + (1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
&\quad = (1-\lambda\alpha)\frac{|ab|P_1}{c}\,{}_3F_2(1+|a|, 1+|b|, 1+P_1; 1+c, 2; 1) + (1-\alpha)\ [{}_3F_2(|a|, |b|, P_1; c, 1; 1) - 1] \\
&\quad \leq 1 - \alpha
\end{aligned}$$

provided the condition (2.11) is satisfied. □

For the choices of $|b| = |a|$ and $b = 1$, we can deduce further corollaries of Theorem 4 and we omit the details involved.

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